Discrete Mathematics 102 (1992) 239–247
North-Holland

Orphan structure of the first-order Reed–Muller codes

Richard A. Brualdi*

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Ning Cai**

Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60680, USA

Vera S. Pless**

Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60680, USA

Received 28 November 1989
Revised 2 April 1990

Abstract

Brualdi, R.A., N. Cai and V.S. Pless, Orphan structure of the first-order Reed–Muller codes, Discrete Mathematics 102 (1991) 239–247.

We investigate a method of combining two codes which we call the outer product. First-order Reed–Muller codes are outer products of a number of copies of the full binary space of length 2, and we apply our results to obtain cosets of the Reed–Muller codes which have no ancestors, that is, which are orphans.

1. Introduction

We extend our investigations reported in [2] of orphans of binary linear codes in general and first-order Reed–Muller codes in particular.

Throughout C denotes a binary linear code of length n. The cosets of C are partially ordered by defining for two cosets C' and C'' of C, $C' \leq C''$ provided there is a coset leader $\boldsymbol{x}'$ of C' and a coset leader $\boldsymbol{x}''$ of C'' such that $\boldsymbol{x}' \leq \boldsymbol{x}''$. Here for vectors $\boldsymbol{x}' = (x_1, \ldots, x_n)$ and $\boldsymbol{x}'' = (x_1'', \ldots, x_n'')$, $\boldsymbol{x}' \leq \boldsymbol{x}''$ means that $x_i'' = 1$ whenever $x_i' = 1$ $(i = 1, \ldots, n)$. The coset C' is a *child* of C'', and C'' is a *parent* of C', provided $C' < C''$ and there is no coset D with $C' < D < D''$. An *orphan* is a coset without any parent. The *covering radius* $\rho = \rho(C)$ of C is the largest weight of a coset, equivalently the largest weight of an orphan. The existence of

* Research partially supported by NSF Grant DMS-8901445 and NSA Grant MDA904-89-H-2060.
** Research partially supported by AFOSR Grant 880237.

orphans of weight less than ρ complicates the determination of the covering radius of a code.

In [2] the orphans of C were characterized under the assumptions that C contains only even weight vectors. We now extend this characterization to include codes with odd weight vectors.

Theorem 1. *Let C' be a coset of C with weight w. Then C' is an orphan if and only if the vectors of C' with weights w and $w+1$ cover all coordinate positions.*

Proof. We first note that each parent of C' is of the form $\boldsymbol{e}_i + C'$ for some unit vector $\boldsymbol{e}_i$ $(1 \leqslant i \leqslant n)$. If the vectors of weights w and $w+1$ of C' cover all coordinate positions, then the weight of $\boldsymbol{e}_i + C'$ is either $w-1$ or w and hence $\boldsymbol{e}_i + C'$ cannot be a parent of C'. Now suppose that C' is an orphan. If there is a coordinate position j which is not covered by any vector of weight w or $w+1$ of C', then $\boldsymbol{e}_j + C'$ contains a vector of weight $w+1$ but contains no vectors of weight w, and it follows that $\boldsymbol{e}_j + C'$ is a parent of C'. $\square$

In the next section we investigate a method of combining two codes C_1 and C_2 which we call the outer product $C_1 \circ C_2$. If C_1 and C_2 are self-complementary, then the outer product of a coset leader of C_1 with a coset leader of C_2 is a coset leader of $C_1 \circ C_2$. There are coset leaders of $C_1 \circ C_2$ which do not arise this way, and as a result the covering radius of $C_1 \circ C_2$ is not readily obtained from the covering radii of C_1 and C_2. The first-order Reed–Muller code $R(1, m)$ is the outer product of m first order Reed–Muller codes $R(1, 1)$. In the third section we find more orphans of $R(1, m)$, some of which arise from the outer product construction.

2. The outer product

Let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be binary vectors of lengths m and n, respectively. The m by n matrix

$$\boldsymbol{x} \circ \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m + y_1 & x_m + y_2 & \cdots & x_m + y_n \end{pmatrix}$$

is called the *outer product* of x and y. (The outer product $x \circ y$ is a special case of a general outer product construction used in linear algebra in which '+' is replaced by an arbitrary binary operation.) Let $\boldsymbol{h}_k$ denote the all 1 vector of length k. We note that:

(i) $\boldsymbol{x} \circ \boldsymbol{y} = \boldsymbol{x} \otimes \boldsymbol{h}_n + \boldsymbol{h}_m \otimes \boldsymbol{y}$,

(ii) $(\boldsymbol{x} + \boldsymbol{x}') \circ (\boldsymbol{y} + \boldsymbol{y}') = \boldsymbol{x} \circ \boldsymbol{y} + \boldsymbol{x}' \circ \boldsymbol{y}'$, and

(iii) $\mathrm{wt}(\boldsymbol{x} \circ \boldsymbol{y}) = \mathrm{wt}(\boldsymbol{x})(n - \mathrm{wt}(\boldsymbol{y})) + (m - \mathrm{wt}(\boldsymbol{x}))\mathrm{wt}(\boldsymbol{y})$.

Let C_1 and C_2 be binary codes of lengths m and n, respectively. We define the *outer product* to be the binary code

$$C_1 \circ C_2 = \{x \circ y : x \in C_1, y \in C_2\},$$

of length mn. It follows from (2) that if C_1 and C_2 are linear codes, then $C_1 \circ C_2$ is also a linear code. It also follows from (3) that if $C_1' = x' + C_1$ and $C_2' = y' + C_2$ are cosets of the linear codes C_1 and C_2, respectively, then $C_1' \circ C_2'$ is a coset of $C_1 \circ C_2$ and

$$C_1' \circ C_2' = x' \circ y' + C_1 \circ C_2.$$

The binary linear code C of length n is *self complementary* provided it contains the all 1's vector h_n. A code C is self-complementary if and only if for each coset C' of C the minimum weight a and maximum weight b of vectors in C' satisfy $a + b = n$. The outer product $C_1 \circ C_2$ is self-complementary if and only if at least one of C_1 and C_2 is.

Proposition 1. *Let C_1 and C_2 be binary linear codes of lengths m and n, respectively. If both C_1 and C_2 are self-complementary, then* $\dim C_1 \circ C_2 = \dim C_1 + \dim C_2 - 1$; *otherwise,* $\dim C_1 \circ C_2 = \dim C_1 + \dim C_2$.

Proof. The map $T : C_1 \oplus C_2 \to C_1 \circ C_2$ defined by $T(\boldsymbol{x} \oplus \boldsymbol{y}) = \boldsymbol{x} \circ \boldsymbol{y}$ is a linear transformation from the direct sum $C_1 \oplus C_2$ onto the outer product $C_1 \circ C_2$. We have $\boldsymbol{x} \circ \boldsymbol{y} = 0$ if and only if $\boldsymbol{x} = 0$ and $\boldsymbol{y} = 0$ or $\boldsymbol{x} = \boldsymbol{h}_m$ and $\boldsymbol{y} = \boldsymbol{h}_n$ and the result follows. □

It follows from Proposition 1 that for cosets C_1' and C_2' of C_1 and C_2, respectively, the map $T : C_1' \oplus C_2' \to C_1' \circ C_2'$ given by $T(\boldsymbol{x} \oplus \boldsymbol{y}) = \boldsymbol{x} \circ \boldsymbol{y}$ is a bijection if one of C_1 and C_2 is not self-complementary and is 2 to 1 otherwise.

We next turn to the calculation of the minimum distance of $C_1 \circ C_2$ and the weights of cosets of the form $C_1' \circ C_2'$. The following equation is a consequence of a straightforward calculation.

Lemma 1. *Let $\boldsymbol{x}$ and $\boldsymbol{x}^*$ be binary m-tuples, and let $\boldsymbol{y}$ and $\boldsymbol{y}^*$ be binary n-tuples. Then*

$$\begin{aligned} \mathrm{wt}(\boldsymbol{x} \circ \boldsymbol{y}) - \mathrm{wt}(\boldsymbol{x}^* \circ \boldsymbol{y}^*) = {} & (\mathrm{wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x}^*))(n - \mathrm{wt}(\boldsymbol{y}) - \mathrm{wt}(\boldsymbol{y}^*)) \\ & + (\mathrm{wt}(\boldsymbol{y}) - \mathrm{wt}(\boldsymbol{y}^*))(m - \mathrm{wt}(\boldsymbol{x}) - \mathrm{wt}(\boldsymbol{x}^*)). \end{aligned} \tag{1}$$

Theorem 2. *Let C_1 and C_2 be binary linear codes of lengths m and n, respectively. Let w_1 and W_1 denote respectively the minimum (nonzero) weight and maximum weight of C_1, and let w_1 and W_2 denote respectively the minimum (nonzero) weight and maximum weight of C_2. Then the minimum weight of a nonzero vector of $C_1 \circ C_2$ equals*

$$\min\{mw_2, w_1 n, W_1(n - W_2) + (m - W_1)W_2\}.$$

Proof. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be arbitrary vectors in C_1 and C_2, respectively. If $\boldsymbol{x}$ is the zero vector of C_1 and $\boldsymbol{y}$ is a nonzero vector of C_2, then $\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) \geqslant mw_2$. If $\boldsymbol{x}$ is a nonzero vector of C_1 and $\boldsymbol{y}$ is the zero vector of C_2, then $\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) \geqslant w_1 n$. Hence we may assume that neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is a zero vector.

Case 1: $\mathrm{wt}(t) \leqslant n - w_2$.

Let $\boldsymbol{y}^*$ be a vector of C_2 with $\mathrm{wt}(\boldsymbol{y}^*) = w_2$ and let $\boldsymbol{x}^* = 0$ denote the zero vector of C_1. Applying Lemma 1 we obtain

$$\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) - mw_2 = \mathrm{wt}(\boldsymbol{x})(n - \mathrm{wt}(\boldsymbol{y}) - w_2) + (\mathrm{wt}(\boldsymbol{y}) - \mathrm{wt}(\boldsymbol{y}^*))(m - \mathrm{wt}(\boldsymbol{x})).$$

Since $\boldsymbol{y}^*$ has minimum weight in C_2, it follows that $\mathrm{wt}(\boldsymbol{y}) \geqslant \mathrm{wt}(\boldsymbol{y}^*)$ and hence $\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) \geqslant mw_2$.

Case 2: $\mathrm{wt}(\boldsymbol{x}) \leqslant m - w_1$.

We use an argument similar to that in Case 1.

Case 3: $\mathrm{wt}(\boldsymbol{x}) > m - w_1$ *and* $\mathrm{wt}(\boldsymbol{y}) > n - w_2$.

We now let $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ be vectors of C_1 and C_2, respectively, with $\mathrm{wt}(\boldsymbol{x}) = W_1$ and $\mathrm{wt}(\boldsymbol{y}^*) = W_2$. Applying Lemma 1 again we obtain

$$\begin{aligned}&\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) - (W_1(n - W_1) + (m - W_1)W_2)\\ &\quad = (\mathrm{wt}(\boldsymbol{x}) - W_1)(n - \mathrm{wt}(\boldsymbol{y}) - W_2) + (\mathrm{wt}(\boldsymbol{y}) - W_2)(m - \mathrm{wt}(\boldsymbol{x}) - W_1).\end{aligned}$$

Since $w_1 \leqslant W_1$ and $w_2 \leqslant W_2$, we have $\mathrm{wt}(\boldsymbol{x}) > m - W_1$ and $\mathrm{wt}(\boldsymbol{y}) > n - W_2$. Hence

$$\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) \geqslant W_1(n - W_2) + (m - W_1)W_2. \qquad \square$$

We note that if both C_1 and C_2 are self-complementary, then by Theorem 2, the minimum weight of a nonzero vector in $C_1 \circ C_2$ equals $\min\{mw_2, w_1 n\}$.

Theorem 3. *Let C_1 and C_2 be self-complementary binary linear codes of lengths m and n, respectively. Let C_1^* and C_2^* be cosets of C_1 and C_2 with coset leaders $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, respectively. Then $\boldsymbol{x}^*\circ\boldsymbol{y}^*$ is a coset leader of the coset $C_1^*\circ C_2^* = \boldsymbol{x}^*\circ\boldsymbol{y}^* + C_1\circ C_2$ of $C_1\circ C_2$, and the weight of $C_1^*\circ C_2^*$ equals* $\mathrm{wt}(\boldsymbol{x}^*)(n - \mathrm{wt}(\boldsymbol{y}^*)) + \mathrm{wt}(\boldsymbol{y}^*)(m - \mathrm{wt}(\boldsymbol{x}^*))$. *Moreover, every coset leader of $C_1^*\circ C_2^*$ can be expressed in the form $\boldsymbol{x}^*\circ\boldsymbol{y}^*$ for some coset leaders $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ of C_1^* and C_2^*, respectively, unless* $\mathrm{wt}(\boldsymbol{x}^*) = m/2$ *or* $\mathrm{wt}(\boldsymbol{y}^*) = n/2$.

Proof. Let $\boldsymbol{x} \in C_1^*$ and $\boldsymbol{y} \in C_2^*$. Since C_1 and C_2 are self-complementary, and $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are cosets leaders, we have $\mathrm{wt}(\boldsymbol{x}) + \mathrm{wt}(\boldsymbol{x}^*) \leqslant m$ and $\mathrm{wt}(\boldsymbol{y}) + \mathrm{wt}(\boldsymbol{y}^*) \leqslant n$. Hence by (1), $\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) \geqslant \mathrm{wt}(\boldsymbol{x}^*\circ\boldsymbol{y}^*)$.

If $\mathrm{wt}(\boldsymbol{x}\circ\boldsymbol{y}) = \mathrm{wt}(\boldsymbol{x}^*\circ\boldsymbol{y}^*)$, then by (1) (i) $\mathrm{wt}(\boldsymbol{x}) = \mathrm{wt}(\boldsymbol{x}^*)$ or $\mathrm{wt}(\boldsymbol{y}^*) = n - \mathrm{wt}(\boldsymbol{y})$ and (ii) $\mathrm{wt}(\boldsymbol{y}) = \mathrm{wt}(\boldsymbol{y}^*)$ or $\mathrm{wt}(\boldsymbol{x}^*) = m - \mathrm{wt}(\boldsymbol{x})$. If $\mathrm{wt}(\boldsymbol{x}) = \mathrm{wt}(\boldsymbol{x}^*)$ and $\mathrm{wt}(\boldsymbol{y}) = \mathrm{wt}(\boldsymbol{y}^*)$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are leaders of their respective cosets. If $\mathrm{wt}(\boldsymbol{y}^*) = n - \mathrm{wt}(\boldsymbol{y})$ and $\mathrm{wt}(\boldsymbol{x}^*) = m - \mathrm{wt}(\boldsymbol{x})$, then $\boldsymbol{x}\circ\boldsymbol{y} = (\boldsymbol{x} + \boldsymbol{h}_m)\circ(\boldsymbol{y} + \boldsymbol{h}_n)$ where $\boldsymbol{x} + \boldsymbol{h}_m$ and $\boldsymbol{y} + \boldsymbol{h}_n$ are coset leaders of C_1^* and C_2^*, respectively. If $\mathrm{wt}(\boldsymbol{x}) = \mathrm{wt}(\boldsymbol{x}^*)$ and $\mathrm{wt}(\boldsymbol{x}^*) =$

m − wt(x), then wt(x^*) = $m/2$. If wt(y) = wt(y^*) and wt(y^*) = n − wt(y), then wt(y^*) = $n/2$. The theorem now follows. □

We remark that if C_1^* is a coset of C_1 with (constant) weight $m/2$, then $C_1^* \circ C_2^*$ is a coset of $C_1 \circ C_2$ of constant weight $mn/2$.

Suppose in Theorem 3, dim $C_1 = k$ and dim $C_2 = l$. By Proposition 1, dim $C_1 \circ C_2 = k + l - 1$. Thus of the $2^{mn-k-l+1}$ cosets of $C_1 \circ C_2$ only $2^{m-k+n-l}$ arise as an outer product $C_1^* \circ C_2^*$ of cosets of C_1 and C_2, respectively. Therefore Theorem 3 seems to give little help in determining the covering radius of $C_1 \circ C_2$ in terms of the covering radii of C_1 and C_2, respectively. Suppose $C_1 = F_2^{(m)}$ and $C_2 = F_2^{(n)}$, the full binary spaces of dimensions m and n, respectively. Then $C_1 \circ C_2$ has dimension equal to $m + n - 1$, and the determination of the covering radius of $C_1 \circ C_2$ is equivalent to the Berlekamp–Gale switching problem (see e.g. [3]).

Let C^* be a coset of the binary code C of length n. Then C^* is *1-covered* provided for each coordinate position i there is a leader of C^* with a 1 in position i. We define *0-covered* in a similar way. It follows from Theorem 1 that a 1-covered coset is necessarily an orphan of C, but the converse does not hold in general unless C has only even weights.

Theorem 4. *Let C_1 and C_2 be self-complementary binary linear codes of lengths m and n, respectively. Let C_1^* be a coset of C_1 and let C_2^* be a coset of C_2. If C_1^* is both 0-covered and 1-covered, then $C_1^* \circ C_2^*$ is a 0-covered and 1-covered coset of $C_1 \circ C_2$ and hence an orphan. If C_1^* is ϵ_1-covered and C_2^* is ϵ_2-covered, then $C_1^* \circ C_2^*$ is $(\epsilon_1 + \epsilon_2)$-covered where $\epsilon_1 + \epsilon_2$ is taken modulo 2.*

Proof. The proofs of all parts of the theorem are similar. We give the proof only in the case that C_1^* is both 0-covered and 1-covered.

Let i and j be integers with $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ be any leader of C_2^*. First suppose that $y_j^* = 1$. By hypothesis there exists a coset leader $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ of C_1^* such that $x_i^* = 0$. Applying Theorem 3 we see that $x^* \circ y^*$ is a coset leader of $C_1^* \circ C_2^*$ with a 1 in position (i, j). If $y_j^* = 0$, we use a leader x^* of C_1^* with $x_i^* = 1$. Hence $C_1^* \circ C_2^*$ is 1-covered and in a similar way one shows that $C_1^* \circ C_2^*$ is 0-covered. □

Suppose in Theorem 4 we take C_2 to be the full binary n-tuple space $F_2^{(n)}$. Then $C_2^* = C_2$ is a 0-covered (but not 1-covered) coset, and it follows that if C_1^* is a 1-covered coset of C_1, then $C_1^* \circ C_2$ is an orphan of $C_1 \circ C_2$. Now let C_2 be the code of all even weight vectors of length n, and let C_2^* be the coset of C_2 different from C_2 itself. Then C_2 has weight 1 and is both 0-covered and 1-covered. Hence $C_1^* \circ C_2^*$ is a 0-covered orphan for every coset C_1^* of C_1.

In the next section we shall use Theorem 4 in order to obtain orphans of first order Reed–Muller codes.

3. Orphans of $R(1, m)$

The first order Reed–Muller code $R(1, 1)$ is the full binary 2-tuple space $F_2^{(2)}$. It follows from the recursive characterization of $R(1, m)$ and the definition of the outer product that

$$R(1, m) = R(1, m-1) \circ F_2^{(2)}.$$

Hence

$$R(1, m) = F_2^{(2)} \circ F_2^{(2)} \circ \cdots \circ F_2^{(2)} \quad (mF_2^{(2)}\text{'s}),$$

a self-complementary code of length 2^m all of whose vectors have even weight. By Theorem 1 a coset of $R(1, m)$ is an orphan if and only if it is 1-covered. By the remarks at the end of the last section, if C_1 is an orphan of $R(1, m-1)$, then $C_1 \circ F_2^{(2)}$ is an orphan of $R(1, m)$. Now $R(1, 2)$ is the code $E^{(4)}$ of all even weight vectors of length 4 and has a unique orphan $E^{(4)*}$ of weight 1. Hence

$$E^{(4)*} \circ F_2^{(2)} \circ \cdots \circ F_2^{(2)} \quad (m-2\, F_2^{(2)}\text{'s})$$

is an orphan of $R(1, m)$ of weight 2^{m-2} $(m \geqslant 2)$. More generally, it is well known that the covering radius of $R(1, l)$ equals $2^{l-1} - 2^{(l-2)/2}$ for l even, and hence $R(1, l)$ has an orphan $\mathrm{O}(1, l)^*$ of weight $2^{l-1} - 2^{(l-2)/2}$ (l even). Hence for $m \geqslant l$,

$$\mathrm{O}(1, l)^* \circ F_2^{(2)} \circ \cdots \circ F_2^{(2)} \quad (m-l\, F_2^{(2)}\text{'s})$$

is an orphan of $R(1, m)$. In this way one obtains $\lceil (m-1)/2 \rceil$ orphans of $R(1, m)$ for $m \geqslant 2$. These are the orphans obtained in [2], but described here using the outer product construction.

Now $E^{(4)*}$ is a 0-covered and 1-covered coset of $R(1, 2)$ of weight 1. Hence by Theorem 4, if C_2^* is any coset of $R(1, m-2)$ of weight α, then

$$E^{(4)*} \circ C_2^*$$

is a 0-covered orphan of $R(1, m)$ of weight $2^{m-2} + 2\alpha$ $(m \geqslant 3)$. Hence by induction we obtain the following.

Theorem 5. *For $m \geqslant 2$, $R(1, m)$ has a 0-covered orphan of weight w for each even integer w satisfying*

$$2^{m-2} \leqslant w \leqslant 2^{m-1} - 2^{\lfloor (m-1)/2 \rfloor}.$$

We note that since the minimum weight of $R(1, m)$ equals 2^{m-1}, no orphan of $R(1, m)$ can have weight less than 2^{m-2}. We also note that if m is even, then $2^{m-1} - 2^{\lfloor (m-1)/2 \rfloor}$ is the covering radius ρ_m of $R(1, m)$ and hence $R(1, m)$ has no orphan of greater weight. If m is odd, then $2^{m-1} - 2^{(m-1)/2}$ is the well known lower bound for the covering radius ρ_m of $R(1, m)$.

Suppose for some odd integer m, the covering radius of $R(1, m)$ exceeded the lower bound $2^{m-1} - 2^{(m-1)/2}$. Then $R(1, m)$ has an orphan C_1^* of weight $w > 2^{m-1} - 2^{(m-1)/2}$. Let l be any odd integer with $l > m$. Then $l - m$ is even and by Theorem 5 $R(1, l-m)$ has a 0-covered orphan C_2^* whose weight equals $2^{l-m-1}-2^{(l-m-2)/2}$. By Theorem 4 $C_1^* \circ C_2^*$ is an orphan of $R(1, l)$ and its weight equals

$$2^{l-1}-2^{(l+m-2)/2}+w2^{(l-m)/2}>2^{l-1}-2^{(l-1)/2}.$$

Thus we have the following.

Corollary 1. *If for some odd integer m, $\rho_m > 2^{m-1}-2^{(m-1)/2}$ then for every odd integer $l > m$, $\rho_l > 2^{l-1}-2^{(l-1)/2}$.*

We remark that in [4] (see also [5]) a computer calculation was used to show that the covering radius of $R(1, 15)$ satisfies

$$\rho_{15} \geqslant 16276 > 2^{14}-2^7.$$

We now show how to obtain recursively all the orphans of $R(1, m)$ of weight 2^{m-2}, $(m \geqslant 3)$ the smallest possible weight of an orphan. We call these orphans the *baby orphans* of $R(1, m)$. In [2] it was shown that the number of baby orphans of $R(1, m)$ equals $\binom{2^m-1}{2}/3$. A baby orphan C_1 of $R(1, m-1)$ has 4 coset leaders $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$ where $\boldsymbol{x}_1+\boldsymbol{x}_2+\boldsymbol{x}_3+\boldsymbol{x}_4$ is the all 1's vector of length 2^{m-1}. Let $\boldsymbol{x}$ be any codeword of $R(1, m-1)$. We have $\boldsymbol{x}+\boldsymbol{x}_j$ as a coset leader of C_1 or the complement of a coset leader if and only if $\boldsymbol{x}+\boldsymbol{x}_i$ is a coset leader or the complement of a coset leader for all $i=1, 2, 3, 4$. It follows that the vectors $(\boldsymbol{x}_i, \boldsymbol{x}_j)$ of length 2^m, $(1 \leqslant i, j \leqslant 4)$ fall into four distinct cosets of $R(1, m)$ and each of these cosets is a baby orphan. Now let $\boldsymbol{x}$ be any of the 2^m-2 codewords of $R(1, m-1)$ of weight 2^{m-2}. Then the vectors $(\boldsymbol{x}, 0)$ fall into $2^{m-2}-1$ cosets of $R(1, m-1)$ ($(\boldsymbol{x}, 0)$ and $(\bar{\boldsymbol{x}}, 0)$ belong to the same coset) of weight 2^{m-2} and each of these is a baby orphan by Theorem 1. Let a_m denote the number of (baby) orphans of $R(1, m)$ obtained recursively in this way starting from $R(1, 2)$. Then $a_2 = 1$ and

$$a_m = 4a_{m-1}+2^{m-2}-1 \quad (m \geqslant 3).$$

It is easy to check that $a_m = \binom{2^m-1}{2}/3$ satisfies the above recurrence relation. Hence we conclude that *all baby orphans of $R(1, m)$ are obtained from the recursive constructions above.* In fact the coset leaders of the baby orphans are exactly the minimum weight vectors of the second order Reed–Muller code $R(2, m)$.

Let $m \geqslant 4$. The baby orphans of $R(1, m-1)$ also give rise to orphans of weight $2^{m-3}+2^{m-2}$ of $R(1, m)$ as follows. Again let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$ denote the leaders of a baby orphan C_1 of $R(1, m-1)$, and let $\boldsymbol{y}_1$ denote a vector of C_1 of weight 2^{m-2} (since $m \geqslant 4$, such vectors $\boldsymbol{y}$ exist [2]). Then $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ is a leader of a coset C_1' of $R(1, m)$ of weight $2^{m-3}+2^{m-2}$ (this is because if $\boldsymbol{y}_1+\boldsymbol{z}$ has weight 2^{m-3} for some codeword $\boldsymbol{z}$ of $R(1, m-1)$, then $\boldsymbol{x}_i+\boldsymbol{z}$ cannot have weight 2^{m-2} or $3 \cdot 2^{m-3}$). The coset C_1' has sixteen leaders, namely,

$$\{(\boldsymbol{x}_i, \boldsymbol{y}_1 + \boldsymbol{x}_i - \boldsymbol{x}_1), (\boldsymbol{x}_i, \boldsymbol{y}_1 + \boldsymbol{x}_i - \boldsymbol{x}_1 + \boldsymbol{h}_{2^{m-1}}) \colon 1 \leqslant i \leqslant 4\},$$

and the vectors obtained by interchanging the order of the two parts of each of these vectors. In particular it follows that C_1' is an orphan of $R(1, m)$. The number of orphans of weight $2^{m-3} + 2^{m-2}$ of $R(1, m)$ obtained in this way equals

$$\frac{\binom{2^{m-1}-1}{2}}{3} \cdot (2^{m-1} - 4),$$

since the number of vectors of weight 2^{m-2} in C_1 equals $2^m - 8$ (see [3]). The weight distribution of these orphans is: $A_{3 \cdot 2^{m-3}} = 16$, $A_{4 \cdot 2^{m-3}} = 2^{m+1} - 32$, $A_{5 \cdot 2^{m-3}} = 16$. Notice that when $m = 4$, these orphans have only the two distinct weights 6 and 10. When $m = 4$, all 28 cosets of $R(1, 4)$ of weight 6 are obtained in this way (6 is the covering radius of $R(1, 4)$).

Now let m be an odd integer with $m \geqslant 3$. Let C_1 and C_2 be cosets of $R(1, m-1)$ with weight equal to the covering radius $\rho_{m-1} = 2^{m-2} - 2^{(m-3)/2}$, and let $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ be cosets leaders of C_1 and C_2, respectively. Then $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ is a leader of a coset C_3 of $R(1, m)$ of weight $2^{m-1} - 2^{(m-1)/2}$ (the known lower bound for the covering radius of $R(1, m)$). Moreover, for each leader $\boldsymbol{x}'$ of C_1, there is a leader $\boldsymbol{y}'$ of C_2 such that $(\boldsymbol{x}', \boldsymbol{y}')$ is a leader of C_3 and for each leader $\boldsymbol{y}'$ of C_2 there is a leader $\boldsymbol{x}'$ of C_1 such that $(\boldsymbol{x}', \boldsymbol{y}')$ is a leader of C_3. Hence it follows from Theorem 1 that C_3 is an orphan of $R(1, m)$. The weight distribution of each coset C_3 obtained in this way is

$$A_{2^{m-1} \pm 2^{(m-1)/2}} = 2^{m-1},$$
$$A_{2^{m-1}} = 2^m.$$

Using the tables in [1] and [6] for the cosets of $R(1, 4)$ and $R(1, 5)$, we identified in [2] all orphans of $R(1, m)$ with $m \leqslant 5$. We conclude this paper by identifying the 0-covered orphans of $R(1, m)$ with $m \leqslant 5$. The code $R(1, 2)$ has covering radius 1. Its unique coset of weight 1 is a baby orphan and it is 0-covered. The code $R(1, 3)$ has covering radius 2. The cosets of weight 2 are the baby orphans and they are 0-covered; no coset of weight 1 is an orphan. Now consider $R(1, 4)$, which has covering radius 6. The only orphans are the cosets of weight 6 and the 35 baby orphans. The baby orphans are all 0-covered. Suppose there is an orphan of weight 6 which is not 0-covered. By [6] this orphan has 16 leaders, and thus it has a descendant coset of weight 5 with 16 leaders, contradicting [6]. Hence all orphans of $R(1, 4)$ are 0-covered. The same type of reasoning shows that the orphans of $R(1, 5)$, as identified in [2] are all 0-covered.

References

[1] E. Berlekamp and L.R. Welch, Weight distributions of the cosets of the (32, 6) Reed–Muller code, IEEE Trans. Inform. Theory 18 (1972) 203–207.

[2] R.A. Brualdi and V.S. Pless, Orphans of the first order Reed–Muller codes, IEEE Trans. Inform. Theory 36 (1990) 399–401.

[3] R.L. Graham and N.J.A. Sloane, On the covering radius of codes, IEEE Trans. Inform. Theory 31 (1985) 385–401.

[4] N.J. Patterson and D.H. Wiedemann, The covering radius of the $(2^{15}, 16)$ Reed–Muller code is at least 16296, IEEE Trans. Inform. Theory 29 (1983) 354–356.

[5] N.J. Patterson and D.H. Wiedemann, Correction to "The covering radius of the $(2^{15}, 16)$ Reed–Muller code is at least 16296", IEEE Trans. Inform. Theory 36 (1990) 443.

[6] N.J.A. Sloane and R.J. Dick, On the enumeration of cosets of first order Reed–Muller codes, in: Proc. IEEE Int. Conf. Communications 7 (1971) 36-2–36-6.